ON THE EVALUATION OF THE CRITICAL REYNOLDS NUMBER FOR THE FLOW OF FLUID BETWEEN TWO ROTATING SPHERICAL SURFACES

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PMM Vol.25, No.5, 1961, pp. 858-866

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(Received April 28, 1961)

The investigation of the onset of flow instability in a closed region has so far been carried out as a nonlinear problem only for the case of fluid motion between two cylinders rotating at different angular velocities [1]. In this problem the equations for the basic laminar motion may be solved exactly for any Reynolds number.

For the flow between two rotating cylinders ("Taylor flow") Taylor has shown theoretically that laminar flow ceases to be stable at a certain critical Reynolds number. The form of the disturbance which broke down the laminar flow was also determined theoretically. Furthermore, Taylor has shown in classical experiments that after breakdown of the basic flow for Reynolds numbers somewhat above the critical value, there appears a new stationary flow whose form is almost indistinguishable from the instability flow resulting from a normal disturbance, and the intensity appears to be proportional to $\sqrt{(R - R_*)}$. As has been shown [2], this latter relationship must apply to unbounded flows, but in regard to the "intensity" of the nonstationary motion arising after the breakdown of the stationary flow, it is evident that the phenomena of the breakdown of stability are of a nonlinear nature. The theoretical investigation of the nonlinear equations of hydrodynamics for the Taylor flow was carried out by Stuart [3], who used a method similar to the method of Landau for unbounded flows. Recently, however, it was discovered [4] that the Taylor problem may not be regarded as a typical closed-flow problem.

In the present paper we consider the stability problem of the fluid flow in the space between two concentric spheres. The problem is solved using the method of small perturbations. From the results of proposed experiments it is expected that the critical Reynolds number will be known. It is assumed that the outer boundary, i.e. the wall of the spherical layer of radius r_2, is stationary, and the inner boundary of radius r_1 is rotating with angular velocity

$$\bar{\Omega} = \Omega \mathbf{n} \qquad (\mathbf{n}^2 = 1) \tag{0.1}$$

Let us choose the following as characteristic quantities: radius of the inner sphere r_1, velocity ν / r_1, moment of forces $r_1, \rho \nu^2$, where ν is the kinematic viscosity, ρ the density of the fluid. Then the Reynolds number is

$$R = r_1^2 \Omega / \nu \tag{0.2}$$

The calculations were carried out for $r_2/r_1 \equiv a = 2$. A solution was looked for in terms of powers of Reynolds number. The convergence of such an expansion for small Reynolds numbers is proved in [5]. Since the calculations become very cumbersome for large powers of R, we had to confine ourselves to terms proportional to R^2. The results obtained by this method, therefore, are not valid for $r_2 >> r_1$ or for $r_2 - r_1 \ll r_1$.

1. Basic laminar flow. The equations of steady motion in terms of the chosen nondimensional quantities have the form

$$(\mathbf{U} \nabla) \mathbf{U} = -\nabla P - \operatorname{rot} \operatorname{rot} \mathbf{U}, \qquad \operatorname{div} \mathbf{U} = 0 \tag{1.1}$$

$\mathbf{U}|_{s_1} = R \mathbf{n} \times \mathbf{r}_1, \qquad \mathbf{U}|_{s_2} = 0$ ($\mathbf{r}_1$ is a unit vector along the radius)

We look for the solutions of these equations in the form of series

$$\mathbf{U} = R\mathbf{U}_1 + R^2 \mathbf{U}_2 + \ldots, \qquad P = RP_1 + R^2 P_2 + \ldots \tag{1.2}$$

The well-known first approximation [2] is

$$\mathbf{U}_1 = \alpha(r)\, \mathbf{n} \times \mathbf{r}, \qquad \alpha(r) = \frac{1}{a^3 - 1}\left(\frac{a^3}{r^3} - 1\right) \tag{1.3}$$

For the second approximation we obtain from (1.1) and (1.2) the equations

$$\nabla P_2 + \operatorname{rot} \operatorname{rot} \mathbf{U}_2 = -(\mathbf{U}_1 \nabla) \mathbf{U}_1, \qquad \operatorname{div} \mathbf{U}_2 = 0, \qquad \mathbf{U}_2|_{s_1, s_2} = 0 \tag{1.4}$$

It is convenient to carry out the solutions of these equations in spherical coordinates r, ϑ, ϕ. We shall denote the coordinate vectors by r_1, ϑ_1 and ϕ_1, and expand the right-hand side in terms of spherical vector functions [6]. The calculation yields

$$(\mathbf{U}_1 \nabla) \mathbf{U}_1 = \alpha^2 r \left[\tfrac{2}{3}(Y_2 - 1)\, \mathbf{r}_1 + \tfrac{1}{3} r \nabla Y_2\right] \qquad \left(Y_2 = \frac{1}{2}(3\cos^2\vartheta - 1)\right) \tag{1.5}$$

Since the operator on the left-hand side of (1.1) is invariant with respect to speed of rotation, the solution must have the form

$$\mathbf{U}_2 = F(r)\,\mathbf{r}_1 Y_2 + G(r)\, r \nabla Y_2$$
$$P_2 = P(r) + Q(r) Y_2 \qquad (1.6)$$

Substitution of Expressions (1.5) and (1.6) into (1.4) gives

$$P' = \tfrac{2}{3} r\alpha^2$$
$$-r^2 Q' - 6F + 6(rG)' = \tfrac{2}{3} r^3 \alpha^2 \qquad (1.7)$$
$$-Q - F' + (rG)'' = \tfrac{1}{3} r^2 \alpha^2$$
$$(r^2 F)' - 6rG = 0$$

$$F(1) = 0, \quad F(a) = 0 \qquad (1.8)$$
$$G(1) = 0, \quad G(a) = 0$$

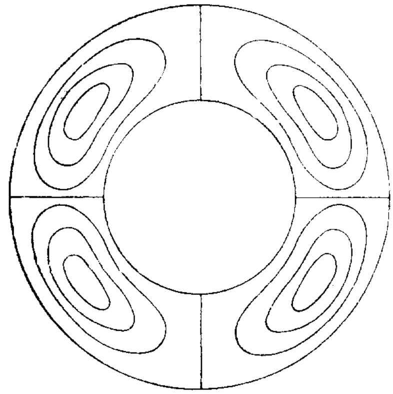

Fig. 1.

To determine function G we eliminate Q and F and obtain by simple calculations

$$F = \frac{a^6}{2(a^3-1)^2}\frac{1}{r^4}\left[-\frac{C_2}{2} + r + C_4 r^2 + \frac{r^4}{a^3} + \frac{C_3 r^5}{3} + \frac{C_1}{5} r^7\right] \qquad (1.9)$$

Four coefficients C are determined from (1.8):

$$C_1 + C_2 + C_3 = 1 - \frac{2}{a^3}, \quad \frac{1}{5}C_1 - \frac{1}{2}C_2 + \frac{1}{3}C_3 + C_4 = -1 - \frac{1}{a^3}$$
$$a^3 C_1 + \frac{1}{a^4}C_2 + aC_3 = -\frac{1}{a^3}, \quad \frac{a^3}{5}C_1 - \frac{1}{2a^4}C_2 + \frac{a}{3}C_3 + \frac{1}{a^2}C_4 = -\frac{2}{a^3} \qquad (1.10)$$

For $a = 2$ we obtain

$$F = \frac{2}{25039 r^4}(r-1)^2(2-r)^2(832 + 452r + 78r^2 + 13r^3) \qquad (1.11)$$

The secondary velocity $\mathbf{U}_2$ superimposed upon the basic flow may be represented in the form

$$\mathbf{U}_2 = \mathrm{rot}\left[-\frac{rF}{6}\mathbf{r} \times \nabla Y_2\right] \qquad (1.12)$$

The meridional "streamlines" for $\mathbf{U}_2$, whose equations are $r^2 F \sin^2\vartheta \cos\vartheta = \mathrm{const}$, are represented in Fig. 1. This secondary motion should never be confused with the motion created by the onset of instability. It exists always, although it may not be noticeable experimentally for small Reynolds numbers, since the ratio of this correction velocity U_2 to U_1 is of the order of one thousandth.

2. Decrements in a stagnant fluid. To investigate the stability we shall find first the decrements of normal perturbations for $R = 0$, i.e. in a stagnant fluid, and, after that we shall calculate their variation with increasing R.

It is known [4] that in a stagnant fluid normal disturbances are attenuated according to the law exp $(-\lambda t)$, where the λ-values are real. For the velocity field of normal perturbations **u** we obtain the equations [4]

$$-\lambda \mathbf{u} + \nabla p + \operatorname{rot}\operatorname{rot}\mathbf{u} = 0, \qquad \operatorname{div}\mathbf{u} = 0, \qquad \mathbf{u}\,|_{S_1,\, S_2} = 0 \tag{2.1}$$

The problem has spherical symmetry, therefore it is convenient to expand the perturbation in terms of spherical vector functions [6]

$$\mathbf{u} = f(r)\,Y\mathbf{r}_1 + g(r)\,r\nabla Y + h(r)\,\mathbf{r}\times\nabla Y, \qquad p = q(r)\,Y \tag{2.2}$$

where $Y \equiv Y_{lm}(\vartheta, \phi) = P_l^{(m)}(\vartheta)\exp(im\phi)$ are spherical functions of order l. Note that $g = h = 0$ for $l = 0$. Because of full spherical symmetry it is clear that the decrements will not depend on the number m (the orientation of the disturbances in space is irrelevant). Therefore, henceforth, in this section $m = 0$ and the index l is dropped. Substituting (2.2) into (2.1) we obtain

$$-\lambda f + p' + \frac{l(l+1)}{r^2}\,[f - (gr)'] = 0 \tag{2.3}$$

$$-\lambda g + \frac{p}{r} + \frac{1}{r}\,[f - (gr)']' = 0$$

$$f' + \frac{2}{r}f - \frac{l(l+1)}{r}g = 0 \tag{2.4}$$

$$\lambda h + h'' + \frac{2}{r}h' - \frac{l(l+1)}{r^2}h = 0$$

where all three functions f, g and h vanish for $r = 1$ and $r = a$. Evidently, the functions h are determined independently of f and q; consequently the perturbations are of two types: for ϕ-disturbances

$$\mathbf{u} = h(r)\,\mathbf{r}\times\nabla Y \tag{2.5}$$

the particles do not cross their spherical layer, whereas for r-disturbances

$$\mathbf{u} = f(r)\,Y\mathbf{r}_1 + g(r)\,r\nabla Y \tag{2.6}$$

the fluid particles have a radial velocity component.

From (2.4) is seen that:

a) The ϕ-perturbation may be defined by Bessel functions of index one half, i.e. by trigonometric functions

$$h(r) = C_1\psi_1(\lambda^{1/2}r) + C_2\psi_2(\lambda^{1/2}r) \tag{2.7}$$

where [7]

$$\psi_1(z) = (-)^l z^l \left(\frac{d}{zdz}\right)^l \left[\frac{\sin z}{z}\right], \qquad \psi_2(z) = z^l \left(\frac{d}{zdz}\right)^l \left[\frac{\cos z}{z}\right] \quad (l = 1, 2, \ldots) \tag{2.8}$$

The boundary conditions yield

$$C_1\psi_1(\lambda^{1/2}) + C_2\psi_2(\lambda^{1/2}) = 0, \qquad C_1\psi_1(\lambda^{1/2}a) + C_2\psi_2(\lambda^{1/2}a) = 0 \tag{2.9}$$

Hence the equation for λ is as follows:

$$\psi_1(\lambda^{1/2})\,\psi_2(\lambda^{1/2}a) = \psi_1(\lambda^{1/2}a)\,\psi_2(\lambda^{1/2}) \tag{2.10}$$

For every l there exists an indefinitely increasing sequence of decrements, and the smallest of them for each l will be henceforth denoted by λ_l.

It is easily shown that $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant, \ldots$. Indeed, the smallest eigenvalue of the problem (2.4) is equal to the minimum of the expression

$$Q_l[h] = \left(\int_1^a [r^2h'^2 + l(l+1)h^2]\,dr\right) \Big/ \left(\int_1^a r^2h^2dr\right) \tag{2.11}$$

if the trial functions vanish at the limits of interval $[1, a]$. Since it is evident that $Q_l[h] < Q_{l+1}[h]$ (the trial functions for all l are the same) then obviously

$$\min Q_l \leqslant \min Q_{l+1} \tag{2.12}$$

Thus, the smallest decrement of the ϕ-perturbation is obtained for $l = 1$. The numerical solution of Equation (2.10) for $a = 2$ yields

$$\lambda_1 = 10.80, \qquad C_2 / C_1 = 2.120, \qquad \mathbf{u} = -h(r)\sin\vartheta\boldsymbol{\varphi}_1 \tag{2.13}$$

b) r-perturbations. In order to solve the system (2.3) we shall first determine the pressure. From (2.1) we have (dropping the index l)

$$\nabla^2 p = p'' + \frac{2}{r}p' - \frac{l(l+1)}{r^2}p = 0 \tag{2.14}$$

Hence

$$p = Ar^l + \frac{B}{r^{l+1}} \qquad (A,\ B = \text{const}) \tag{2.15}$$

Then, eliminating g from (2.3), we obtain

$$f'' + \frac{4}{r} f' + \left[\lambda - \frac{l(l+1)-2}{r^2}\right] f = lAr^{l-1} - \frac{(l+1)B}{r^{l+2}} \tag{2.16}$$

with the boundary conditions

$$f(1) = f(a) = f'(1) = f'(a) = 0 \tag{2.17}$$

The homogeneous equation, corresponding to (2.16), has the solution

$$\frac{1}{r}[D_1\psi_1(\lambda^{1/2}r) + D_2\psi_2(\lambda^{1/2}r)] \tag{2.18}$$

and Equation (2.16) is easily solved by the method of variation of coefficients. Let us introduce the notation

$$I_1(r, \lambda) = \int_1^r r^{l+2}\psi_1(\lambda^{1/2}r)\,dr, \qquad I_3(r, \lambda) = \int_1^r r^{-l+1}\psi_1(\lambda^{1/2}r)\,dr$$

$$I_2(r, \lambda) = \int_1^r r^{l+2}\psi_2(\lambda^{1/2}r)\,dr, \qquad I_4(r, \lambda) = \int_1^r r^{-l+1}\psi_2(\lambda^{1/2}r)\,dr \tag{2.19}$$

Integrals (2.19) are calculated directly, using known formulas [8]. Then the exact solution of (2.16), satisfying conditions (2.17) at the lower limit, will be

$$f(r) = \frac{A}{r}(\psi_1 I_2 - \psi_2 I_1) + \frac{B}{r}(\psi_1 I_4 - \psi_2 I_3) \tag{2.20}$$

Conditions (2.17) for $r = a$ lead to the equations

$$A[\psi_1(\lambda^{1/2}a) I_2(\lambda, a) - \psi_2(\lambda^{1/2}a) I_1(\lambda, a)] + B[\psi_1(\lambda^{1/2}a) I_4(\lambda, a) - \psi_2(\lambda^{1/2}a) I_3(\lambda, a)] = 0$$

$$A[\psi_1'(\lambda^{1/2}a) I_2(\lambda, a) - \psi_2'(\lambda^{1/2}a) I_1(\lambda, a)] + B[\psi_1'(\lambda^{1/2}a) I_4(\lambda, a) - \psi_2'(\lambda^{1/2}a) I_3(\lambda, a)] = 0$$

where the prime signifies derivatives with respect to r. Equating to zero the determinant of this system, we obtain

$$I_1(\lambda, a) I_4(\lambda, a) = I_2(\lambda, a) I_3(\lambda, a) \tag{2.21}$$

This condition determines the spectrum of λ-decrements. In practice, it is impossible to calculate the variation of all the decrements with R. Therefore, at first the smallest decrements for various l were calculated, then those among them were selected which, according to physical considerations, must decrease faster with increase of R than the remainder. For $a = 2$ the smallest decrements for $l = 1, 2, 3, \ldots$

are equal to $\lambda_1 = 38.62$, $\lambda_2 = 37.49$, $\lambda_3 = 36.9$, $\lambda_4 = 37.2$, $\lambda_5 = 38.9$. Morever they sharply increase with increase of l.

It is most probable that the laminar flow will be broken down through the perturbations, whose lines of flow coincide approximately with streamlines of the secondary velocity of $\mathbf{U}_2$ of the basic flow. This condition is satisfied by the perturbation in the case $l = 2$. Precisely for this case the calculations were carried out which verify the assumptions. For comparison the calculations were carried out also for $l = 1$.

Since Formula (2.20) is cumbersome for numerical calculations, it is desirable to obtain approximate expressions for $f_1(r)$ and $f_2(r)$. To do this we reduce Equations (2.3) to a variational problem. Eliminating p and q, we obtain

$$(r^4f'')'' - 2s\,(r^2f')' + s\,(s-2)\,f = \lambda\,[(s-2)\,r^2f - (r^4f')'] \quad (2.22)$$

where for simplification $s = l\,(l + 1)$. The smallest eigenvalue λ_1 is equal to the minimum of expression

$$\lambda_l\,(s) = \int_1^a [r^4f''^2 + 2sr^2f'^2 + s\,(s-2)\,f^2]\,dr \Big/ \int_1^a [r^4f'^2 + (s-2)r^2f^2]\,dr \quad (2.23)$$

with the condition (2.17). Hence, using the Ritz method, we obtain for $a = 2$

$$f_1\,(r) = A_1\,(r-1)^2\,(2-r)^2\,(1 - 0.446\,r),$$
$$f_2\,(r) = A_2\,(r-1)^2\,(2-r)^2\,(1 - 0.426\,r) \quad (2.24)$$

The constants $A_1 = 6.39$ and $A_2 = 12.47$ for $a = 2$ were determined from the condition of normalization

$$\int_V \mathbf{u}^2 dV = 1 \qquad (V \text{ is the volume of fluid}) \quad (2.25)$$

3. Decrements in the case of slow flow. For small R's normal perturbations may be expanded in series (the index l is dropped)

$$\mathbf{u} + \mathbf{u}_1R + \mathbf{u}_2R^2 + \ldots \quad (3.1)$$

For the velocity field $\mathbf{u}_1$ we obtain the system of equations [4]

$$-\lambda\mathbf{u}_1 + \nabla p_1 + \operatorname{rot}\operatorname{rot}\mathbf{u}_1 = -[(\mathbf{U}_1\nabla)\,\mathbf{u} + (\mathbf{u}\nabla)\,\mathbf{U}_1] \quad (3.2)$$
$$\operatorname{div}\mathbf{u}_1 = 0, \qquad u_1|_s = 0$$

a) Treatment of the ϕ-perturbations. Using (1.3) and (2.13), let us expand the right-hand side of (3.2) in terms of spherical vector functions

$$(\mathbf{U}_1\nabla)\,\mathbf{u} + (\mathbf{u}\nabla)\,\mathbf{U}_1 = -2ah\sin\vartheta\,\mathbf{n}\times\varphi_1 = \frac{2h\alpha}{3}\,[2\,(1-Y_2)\,\mathbf{r}_1 - r\nabla Y_2] \quad (3.3)$$

In agreement with (3.3) we are looking for the solution of (3.2) in the form

$$\mathbf{u}_1 = \beta\,(r)\,Y_2\,\mathbf{r}_1 + \gamma\,(r)\,r\nabla Y_2, \qquad p_1 = q_0 + q_2Y_2 \quad (3.4)$$

Projecting (3.2) on the axes of spherical coordinates, we find

$$-\frac{4h\alpha}{r} = q_0', \qquad \frac{4h\alpha}{3} = q_2' + \frac{6\beta}{r^2} - \frac{6\,(\gamma r)'}{r^2} - \lambda_1\beta, \quad \frac{2h\alpha}{3} = \frac{q_2}{r} + \frac{\beta'}{r} - \frac{(\gamma r)''}{r} - \lambda_1\gamma \quad (3.5)$$

The equation of continuity is

$$\beta' + \frac{2\beta}{r} - \frac{6\gamma}{r} = 0 \quad (3.6)$$

The boundary conditions are

$$\beta\,(1) = \beta\,(a) = \gamma\,(1) = \gamma\,(a) = 0 \quad (3.7)$$

To determine functions β and γ let us first find q_2. To do this we take the divergence of both parts of (3.2) and equate the coefficients of the spherical function Y_2 on the left- and the right-hand sides of the equation. After some calculations we obtain

$$q_2 = \frac{4}{3r^3}\,Z\,(r) + \frac{C}{2}\,r^2 + \frac{D}{3r^3} \qquad \left(Z\,(r) \equiv \int h\alpha r^3\,dr\right) \quad (3.8)$$

Eliminating the function γ from (3.5), we obtain the equation for

$$\beta'' + \frac{4}{r}\,\beta' + \left(\lambda_1 - \frac{4}{r^2}\right)\beta = -\frac{4}{r^4}\,Z + Cr - \frac{D}{r^4}$$
$$\beta\,(1) = \beta\,(a) = \beta'\,(1) = \beta'\,(a) = 0 \quad (3.9)$$

The solution of the homogeneous equation, in agreement with (3.9), is

$$r^{-3/2}\,[a_1J_{5/2}\,(\lambda_1^{\,1/2}r) + a_2J_{-5/2}(\lambda_1^{\,1/2}r)]$$

where $J_p(x)$ is a Bessel function of the first kind of order p.

TABLE

r	Φ_1	Φ_2	Φ_3	β
1.0	0	0	0	0
1.1	+0.01718	−0.01312	+0.00541	+0.000103
1.2	+0.02522	−0.01827	+0.00759	+0.000283
1.3	+0.02258	−0.01661	+0.00631	+0.000341
1.4	+0.01237	−0.00840	+0.00237	+0.000279
1.5	−0.00072	+0.00202	−0.00222	+0.000161
1.6	−0.01146	+0.01075	−0.00606	+0.000034
1.7	−0.01682	+0.01495	−0.00765	−0.000037
1.8	−0.01562	+0.01364	−0.00679	−0.000040
1.9	−0.00872	+0.00788	−0.00409	−0.000033
2.0	0	0	0	0

By varying coefficients, the solution of (3.9) is easily found so as to satisfy the conditions $\beta(1) = \beta'(1) = 0$:

$$\beta(r) = 2\pi r^{-3/2}\Big\{J_{5/2}(\lambda_1^{1/2}r)\int_1^r r^{-3/2}J_{-5/2}(\lambda_1^{1/2}r)\,Zdr -$$

$$- J_{-5/2}(\lambda_1^{1/2}r)\int_1^r r^{-3/2}J_{5/2}(\lambda_1^{1/2}r)\,Zdr +$$

$$+\frac{C}{4\lambda_1^{1/2}}\left[\frac{2r^{5/2}}{\pi\lambda_1^{1/2}}+J_{5/2}(\lambda_1^{1/2}r)\,J_{-7/2}(\lambda_1^{1/2})+J_{-5/2}(\lambda_1^{1/2}r)\,J_{7/2}(\lambda_1^{1/2})\right]+$$

$$+\frac{D}{4\lambda_1^{1/2}}\left[-\frac{2r^{-5/2}}{\pi\lambda_1^{1/2}}+J_{5/2}(\lambda_1^{1/2}r)\,J_{-3/2}(\lambda_1^{1/2})+J_{-5/2}(\lambda_1^{1/2}r)\,J_{3/2}(\lambda_1^{1/2})\Big\}\right] \quad (3.10)$$

The function y is determined from (3.6). The constants C and D in (3.10) have to be found from conditions $\beta(a) = \beta'(a) = 0$. For $a = 2$, numerical calculations yield $C = -0.006374$ and $D = -0.4531$. The values of β are represented in the table.

From the equations of the second correction to the perturbation [4] we have

$$-\lambda_1\mathbf{u}_2 - \lambda_1^{(2)}\mathbf{u} + \nabla p_2 + \text{rot rot } \mathbf{u}_2$$
$$= - [(\mathbf{U}_1\nabla)\,\mathbf{u} + (\mathbf{u}\nabla)\,\mathbf{U}_2 + (\mathbf{u}_1\nabla)\,\mathbf{U}_1 + (\mathbf{U}_1\nabla)\,\mathbf{u}_1] \quad (3.11)$$

Let us determine the correction to the decrement. For this purpose we multiply (3.11) by $\mathbf{u}$ and integrate over the whole volume V of the liquid

$$\lambda_1^{(2)} = \int_V \mathbf{u}\,(\mathbf{u}\nabla)\,\mathbf{U}_2\,dV + \int_V \mathbf{u}\,(\mathbf{u}_1\nabla)\,\mathbf{U}_1\,dV + \int_V \mathbf{u}\,(\mathbf{U}_1\nabla)\,\mathbf{u}_1 dV \quad (3.12)$$

with the condition

$$\int_V \mathbf{u}^2 \, dV = 1$$

After integrating over the angles of (3.12) we obtain

$$\lambda_1^{(2)} = \frac{8\pi}{15}\left\{-\int_1^a (F + 3G)\, h^2 r dr + \int_1^a (\beta + 3\gamma)\, h a r^2 dr + \right.$$

$$\left. + \int_1^a [\beta\ (ar)' + 3\gamma\alpha]\ h r^2 dr\right\} \tag{3.13}$$

Numerical integration for $a = 2$ yields $\lambda_1^{(2)} = -\ 0.000212$. The decrement of the ϕ-perturbation, to the R^2 approximation, is

$$\lambda_1 = 10.80\left[1 - \left(\frac{R}{226}\right)^2\right] \tag{3.14}$$

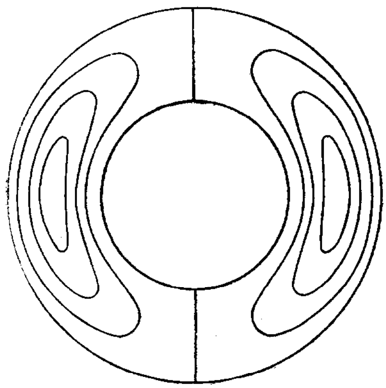

Fig. 2.

b) Treatment of the r-perturbation. 1. Perturbation case $l = 1$. Its meridional streamlines are defined by the equation (Fig. 2)

$$f_1 r^2 \sin^2 \vartheta = \text{const} \tag{3.15}$$

This perturbation opposes the basic flow in some regions of the cavity (Fig. 1).

We shall expand the right-hand side of (3.2) in terms of spherical vector functions

$$(\mathbf{U}_1\nabla)\,\mathbf{u} + (\mathbf{u}\nabla)\,\mathbf{U}_1 = (f_1 r\alpha' + 2f_1\alpha - 2g_1\alpha)\sin\vartheta\cos\vartheta\varphi_1 =$$

$$= \frac{1}{3}\,(f_1'\alpha r - f_1\alpha' r)\ \mathbf{r}\times\nabla Y_2 \tag{3.16}$$

Let us look for $\mathbf{u}_1$ satisfying this function in the form

$$u_1 = \Phi_2(\mathbf{r})\ r \times \nabla Y_2 \tag{3.17}$$

Projecting (3.2) on the axis ϕ_1, we obtain

$$\Phi_2'' + \frac{2}{r}\,\Phi_2' + \left(\lambda_1 - \frac{6}{r^2}\right)\Phi_2 = \frac{1}{3}\,(f_1'\alpha r - f_1 r\alpha') \tag{3.18}$$

This equation was integrated numerically. The function Φ_2 for $a = 2$

is represented in the table. After integrating over the angles in (3.12) we obtain in this case

$$\lambda_1^{(2)} = \frac{8\pi}{15}\int_1^a [3\Phi_2 (f_1 - g_1)\,\alpha + f_1^2 F' + 3\,(F - G)\frac{f_1 g_1}{r} + $$
$$+ 3 f_1 g_1 G' + \frac{g_1^2}{r}(9G - F)]\; r^2 dr \qquad (3.19)$$

under the condition of (2.25). Numerical integration for $a = 2$ yields $\lambda_1^{(2)} = 0.00658$. The decrement of r-disturbance for $l = 1$ approximated to the R^2 term is

$$\lambda_1 = 38.6\left[1 + \left(\frac{R}{77}\right)^2\right] \qquad (3.20)$$

Thus, the larger R, the stronger the damping of this perturbation. The case when $l = 3$ is analogous to this case and the calculations were omitted.

2. Perturbation case $l = 2$. This case is unique because its streamlines given in the meridional plane by the equation

$$f_2 r^2 \cos\vartheta \sin^2\vartheta = \mathrm{const} \qquad (3.21)$$

are parallel to the streamlines of the secondary flow imposed upon the basic flow and are almost indistinguishable from them (Fig. 1). Therefore, we may assume offhand, that this perturbation is most apt to break down the laminar flow. We shall expand the right-hand side of (3.2) in terms of spherical vector functions

$$(3.22)$$
$$(\mathbf{U}_1\nabla)\mathbf{u} + (\mathbf{u}\nabla\mathbf{U}_1 = \left[\frac{3\cos^2\vartheta - 1}{2}\left(\alpha' f_2 + 2\,\frac{\alpha f_2}{r}\right) - 6\cos^2\vartheta\,\frac{\alpha g_2}{r}\right]\mathbf{n}\times\mathbf{r} =$$
$$= \frac{r}{5}\left(f_2'\alpha + f_2\alpha' + \frac{4 f_2\alpha}{r}\right)\mathbf{r}\times\nabla Y_1 + \frac{r}{5}\left(\frac{2}{3} f_2'\alpha - f_2\alpha' - \frac{2}{3}\frac{f_2\alpha}{r}\right)\mathbf{r}\times\nabla Y_3$$

Let us look for a solution of (3.2) satisfying the above in the form

$$\mathbf{u}_1 = \Phi_1(r)\,\mathbf{r}\times\nabla Y_1 + \Phi_3(r)\,\mathbf{r}\times\nabla Y_3 \qquad (3.23)$$

Projecting (3.2) upon the ϕ_1-axis and equating the coefficients of the same spherical functions we obtain

$$\Phi_1'' + \frac{2}{r}\Phi_1' + \left(\lambda_2 - \frac{2}{r^2}\right)\Phi_1 = -\frac{r}{5}\left(f_2'\alpha + f_2\alpha' + \frac{4 f_2\alpha}{r}\right)$$
$$\Phi_3'' + \frac{2}{r}\Phi_3' + \left(\lambda_2 - \frac{12}{r^2}\right)\Phi_3 = -\frac{r}{5}\left(\frac{2}{3} f_2'\alpha - f_2\alpha' - \frac{2}{3}\frac{f_2\alpha}{r}\right)$$

These equations were integrated numerically. The values of functions Φ_1 and Φ_2 for $a = 2$ are represented in the table.

After integrating over the angles in (3.12) we obtain

$$\lambda_2^{(2)} = \frac{8\pi}{35}\int_1^u \Big\{ f_2\alpha\left(6\Phi_3 - \frac{7}{3}\Phi_1\right) + g_2\alpha\,(12\Phi_3 - 7\Phi_1) +$$
$$+\left[f_2^2F' + 9\,\frac{g_2}{r}G + 3\left(f_2g_2G' + \frac{f_2g_2F}{r} + \frac{g_2^2F}{r} - \frac{f_2g_2G}{r}\right)\right]\Big\}\, r^2dr$$

The numerical integration for $a = 2$ yields $\lambda_2^{(2)} = -0.00367$, with the condition of (2.25). The decrement λ_2 for the r-perturbation in the R^2 approximation is

$$\lambda_2 = 37.49\left[1 - \left(\frac{R}{101}\right)^2\right] \qquad (3.24)$$

4. Conclusions. The behavior of decrements with increase of R is represented in Fig. 3, where curve 1 is the ϕ-perturbation for $l = 1$, curve 2 is the r-perturbation for $l = 2$ and curve 3 is the r-perturbation for $l = 1$. The decrement λ_1 for the r-perturbation increases with R; consequently, the basic flow is stable with respect to this perturbation. The two other curves slope down, however; curve 2 intersects the x-axis at $R_* \approx 100$, likewise curve 1 at $R_*' \approx 230$. This means that the laminar flow

$$\mathbf{U} = (\alpha\mathbf{n}\times\mathbf{r})R + (FY_2\mathbf{r}_1 + Gr\nabla Y_2)\,R^2 \qquad (4.1)$$

will be broken down by the r-perturbation in the case $l = 2$

$$\mathbf{u} = (f_2Y_2\mathbf{r}_1 + g_2r\nabla Y_2) + (\Phi_1\mathbf{r}\times\nabla Y_1 + \Phi_3\mathbf{r}\times\nabla Y_3)\,R \qquad (4.2)$$

for Reynolds numbers of the order 100.

Qualitatively, flows (4.1) and (4.2) do not differ from each other. In both cases there is a velocity component along ϕ_1, proportional to R. The meridional streamlines are also similar in the case of both motions. In this manner, after the breakdown of the basic flow, a new flow will be established which is almost of the same form. As established by the nonlinear theory [4], the new stationary motion will be

$$\mathbf{U} + b\mathbf{u}_2\ \ (R - R_*) \qquad (4.3)$$

It will be observed only for $R > R_*$, where it is stable. Although the coefficients b were not calculated, it may be surmised that the secondary flow for $R > R_*$ will be directed into the same direction as the basic flow. For $R < R_*$ it is in the opposite direction and it is not stable.

The observations of these phenomena will be very difficult, because

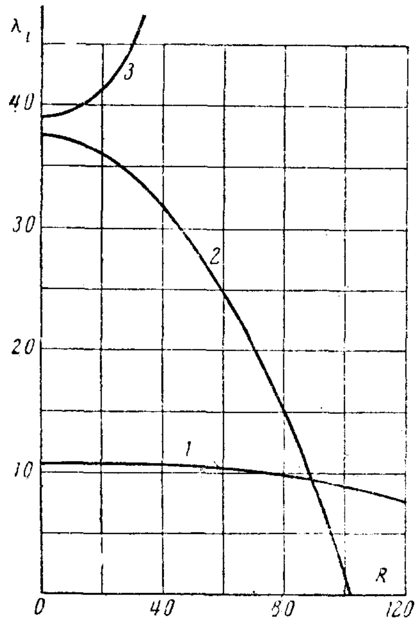

Fig. 3.

the appearance of the new motion may be detected only by a break in the curve which determines the intensity of the flow as a function of R.

I take this opportunity to thank V.S. Sorokin for suggesting the problem and for his valuable help.

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Translated by J.R.W.